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On the strong stability of finite difference schemes for hyperbolic systems in two space dimensions

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Abstract

We study the stability of some finite difference schemes for symmetric hyperbolic systems in two space dimensions. For the so-called upwind scheme and the Lax-Wendroff scheme with a stabilizer, we show that stability is equivalent to *strong stability*, meaning that both schemes are either unstable or ℓ^2 -decreasing. These results improve on a series of partial results on strong stability. We also show that, for the Lax-Wendroff scheme without stabilizer, strong stability may not occur no matter how small the CFL parameters are chosen. This partially invalidates some of Turkel's conjectures in [12].

AMS subject classification: 65M12, 65M06, 35L45.

Keywords: Hyperbolic systems, finite difference schemes, stability.

1 Introduction

Finite difference schemes are commonly used to approximate solutions to hyperbolic systems of conservation laws. In this article, we are interested in the stability of such finite difference schemes when applied to constant coefficients symmetric hyperbolic systems in two space dimensions. Symmetry is often crucial to show stability for a finite difference scheme, see for instance [16], and we shall therefore restrict to this framework.

For linear schemes, stability can be analyzed by means of Fourier transform and is found to be equivalent to uniform power boundedness of the so-called amplification matrix, see for instance [6, chapter 6]. The latter condition reads:

$$\sup_{n \in \mathbb{N}} \sup_{(\xi, \eta) \in \mathbb{R}^2} \|C(\xi, \eta)^n\| < +\infty,$$

where we use from now on the notation (ξ, η) for the frequencies associated with the space variables $(x, y) \in \mathbb{R}^2$, and $\|\cdot\|$ denotes the spectral norm associated with the Hermitian norm for vectors:

$$\forall M \in \mathcal{M}_d(\mathbb{C}), \quad \|M\| = \sup_{X \in \mathbb{C}^d, |X|=1} |MX|, \quad |X|^2 := |X_1|^2 + \cdots + |X_d|^2.$$

One crucial ingredient in the analysis is the fact that the amplification matrix $C(\xi, \eta)$ is a complex *symmetric* matrix, which simplifies the computation of its norm and/or its numerical radius.

Two main subclasses of numerical schemes occur in practice:

1. *Strongly stable* schemes, for which the amplification matrix satisfies $\|C(\xi, \eta)\| \leq 1$ for all $(\xi, \eta) \in \mathbb{R}^2$. This terminology dates back (at least) to [10]. These are exactly the schemes for which the ℓ^2 -norm of the discrete solution decreases at each time step.
2. Schemes for which the numerical radius of $C(\xi, \eta)$ is not larger than 1 for all (ξ, η) . This property yields the bound

$$\sup_{n \in \mathbb{N}} \sup_{(\xi, \eta) \in \mathbb{R}^2} \|C(\xi, \eta)^n\| \leq 2.$$

The numerical radius is a very efficient tool for proving stability in specific situations, and this technique goes back to the seminal work [8]. The reader is also referred to the nice review [5] for a detailed introduction with further references. Strong stability is further studied in [11] where an equivalent condition is found in terms of real vectors. The latter condition is applied to the Lax-Wendroff scheme with stabilizer, and we shall refine some of these results below.

The main question we ask in this article is : does there exist a symmetric hyperbolic system with a numerical scheme that belongs to the second class without being strongly stable ? This seems far from obvious since as time goes by, one sometimes ends up showing that stability is in fact *equivalent* to strong stability. This is what we prove below for the so-called Lax-Wendroff scheme with stabilizer introduced in [8]. Even though strong stability may not be always equivalent to stability, it might still be recovered for sufficiently small CFL parameters. Quoting [12, page 128]: "It is interesting to speculate whether one can strengthen the basic conjecture by even claiming that when a symmetric hyperbolic scheme is stable, it is powerbounded with constant 2 and for sufficiently small $\Delta t/\Delta x$ even strongly stable." This is unfortunately not true, as detailed on a specific example below. To the best of our knowledge, this seems to be the first example of a numerical scheme of the second class that is shown not to be strongly stable. Still, our counterexample is not dissipative in the sense of Kreiss [7]. In this more restrictive framework, we shall show that strong stability can be recovered for sufficiently small CFL parameters.

Let us conclude by observing that strong stability is also a powerful tool in the theory of discretized initial boundary value problems, see [3]. We plan to investigate the influence of the absence of strong stability on semigroup estimates in a near future (this was the main motivation for studying the gap between the above two classes).

2 Main results

We consider a symmetric hyperbolic system in two space dimensions:

$$\begin{cases} \partial_t u + A_\# \partial_x u + B_\# \partial_y u = 0, & t \geq 0, (x, y) \in \mathbb{R}^2, \\ u|_{t=0} = u_0. \end{cases} \quad (1)$$

The matrices $A_\#, B_\#$ belong to $\mathcal{M}_d(\mathbb{R})$ and are symmetric, so that the Cauchy problem (1) is well-posed in $L^2(\mathbb{R}^2)$, see e.g. [2]. Moreover, the solution to (1) satisfies

$$\forall t \geq 0, \quad \|u(t)\|_{L^2(\mathbb{R}^2)} = \|u_0\|_{L^2(\mathbb{R}^2)}.$$

We introduce a finite difference approximation of (1). Let $\Delta x > 0$ and $\Delta y > 0$ denote some space steps in the x and y directions, and let Δt denote the time step. Then the vector $u_{j,k}^n$, where $(n, j, k) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{Z}$, denotes an approximation of $u(n \Delta t, j \Delta x, k \Delta y)$. We define the CFL parameters

$$\lambda := \frac{\Delta t}{\Delta x}, \quad \mu := \frac{\Delta t}{\Delta y},$$

and for later use, we define the matrices

$$A := \lambda A_{\sharp}, \quad B := \mu B_{\sharp}.$$

We refer to [4, chapter IV.3], and [6, chapter 6] for a general description of finite difference schemes for two-dimensional hyperbolic systems, and we shall thus assume that the reader is familiar with the basic ℓ^2 -stability theory of finite difference schemes. In this article, we shall study the stability of three specific schemes:

- The upwind scheme:

$$\begin{aligned} u_{j,k}^{n+1} = & u_{j,k}^n - \frac{1}{2} A (u_{j+1,k}^n - u_{j-1,k}^n) - \frac{1}{2} |A| (2 u_{j,k}^n - u_{j+1,k}^n - u_{j-1,k}^n) \\ & - \frac{1}{2} B (u_{j,k+1}^n - u_{j,k-1}^n) - \frac{1}{2} |B| (2 u_{j,k}^n - u_{j,k+1}^n - u_{j,k-1}^n). \end{aligned} \quad (2)$$

- The Lax-Wendroff scheme with stabilizer (see [8]):

$$\begin{aligned} u_{j,k}^{n+1} = & u_{j,k}^n - \frac{1}{2} A (u_{j+1,k}^n - u_{j-1,k}^n) - \frac{1}{2} B (u_{j,k+1}^n - u_{j,k-1}^n) \\ & - \frac{1}{2} A^2 (2 u_{j,k}^n - u_{j+1,k}^n - u_{j-1,k}^n) - \frac{1}{2} B^2 (2 u_{j,k}^n - u_{j,k+1}^n - u_{j,k-1}^n) \\ & + \frac{1}{8} (A B + B A) (u_{j+1,k+1}^n - u_{j+1,k-1}^n - u_{j-1,k+1}^n + u_{j-1,k-1}^n) \\ & - \frac{1}{8} (A^2 + B^2) (4 u_{j,k}^n - 2 u_{j,k+1}^n - 2 u_{j,k-1}^n \\ & - 2 u_{j+1,k}^n + u_{j+1,k+1}^n + u_{j+1,k-1}^n - 2 u_{j-1,k}^n + u_{j-1,k+1}^n + u_{j-1,k-1}^n). \end{aligned} \quad (3)$$

- The Lax-Wendroff scheme without stabilizer (see again [8]):

$$\begin{aligned} u_{j,k}^{n+1} = & u_{j,k}^n - \frac{1}{2} A (u_{j+1,k}^n - u_{j-1,k}^n) - \frac{1}{2} B (u_{j,k+1}^n - u_{j,k-1}^n) \\ & - \frac{1}{2} A^2 (2 u_{j,k}^n - u_{j+1,k}^n - u_{j-1,k}^n) - \frac{1}{2} B^2 (2 u_{j,k}^n - u_{j,k+1}^n - u_{j,k-1}^n) \\ & + \frac{1}{8} (A B + B A) (u_{j+1,k+1}^n - u_{j+1,k-1}^n - u_{j-1,k+1}^n + u_{j-1,k-1}^n). \end{aligned} \quad (4)$$

We recall that in (2) the matrices $|A|, |B|$ are defined as follows: let P, Q denote orthogonal matrices that diagonalize A and B :

$$P^{-1} A P = \text{diag} (\alpha_1, \dots, \alpha_d), \quad Q^{-1} B Q = \text{diag} (\beta_1, \dots, \beta_d).$$

Then the matrices $|A|$, and $|B|$, are given by:

$$P^{-1} |A| P := \text{diag} (|\alpha_1|, \dots, |\alpha_d|), \quad Q^{-1} |B| Q = \text{diag} (|\beta_1|, \dots, |\beta_d|).$$

Our main results are the following:

Theorem 1. • *The scheme (2) is stable in $\ell^2(\mathbb{Z}^2)$ if and only if*

$$\forall X \in \mathbb{R}^d, \quad \lambda \langle X, |A_{\sharp}| X \rangle + \mu \langle X, |B_{\sharp}| X \rangle \leq |X|^2,$$

and in that case, it is even strongly stable.

- The scheme (3) is stable in $\ell^2(\mathbb{Z}^2)$ if and only if

$$\forall X \in \mathbb{R}^d, \quad \lambda^2 |A_{\sharp} X|^2 + \mu^2 |B_{\sharp} X|^2 \leq \frac{1}{2} |X|^2,$$

and in that case, it is even strongly stable.

The scheme (2) is a particular case of the finite volume scheme studied in [14]. It is obtained by choosing rectangles $[j\Delta x_1, (j+1)\Delta x_1] \times [k\Delta x_2, (k+1)\Delta x_2]$ as control volumes. For such control volumes, the stability condition obtained in [14] reads:

$$\Delta t \frac{2(\Delta x + \Delta y)}{\Delta x \Delta y} \max(\|A_{\sharp}\|, \|B_{\sharp}\|) \leq 1,$$

or equivalently

$$(\lambda + \mu) \max(\|A_{\sharp}\|, \|B_{\sharp}\|) \leq \frac{1}{2}.$$

Our sufficient stability condition in Theorem 1 is less restrictive, but is restricted to uniform cartesian grids while the condition in [14] applies to general triangulations.

The second part of Theorem 1 improves earlier criteria for the strong stability of (3). In [8], Lax and Wendroff prove stability for (3) under the assumption of Theorem 1 by means of the numerical radius. Abarbanel and Gottlieb [1] show *strong stability* under the assumption $\max(\|A\|, \|B\|) \leq 1/2$ and, later on, Tadmor [11] shows strong stability under the assumption $|Au|^4 + |Bu|^4 \leq 1/8$ for all unit vector u . Theorem 1 gives a final optimal version to the stability analysis of (3).

In view of Theorem 1 and other known results in the literature, see e.g. [13, 15, 17] for schemes in which the amplification matrix is either normal or a product of normal matrices, it might seem that strong stability is a common feature of many numerical schemes. The scheme (4) seems to be an exception for no strong stability result has been obtained so far in the case of noncommuting matrices. This is explained by our second main result.

Theorem 2. *Let A_{\sharp} and B_{\sharp} be given by*

$$A_{\sharp} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_{\sharp} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the scheme (4) is never strongly stable, though it is stable if $\lambda^{2/3} + \mu^{2/3} \leq 1$.

If A_{\sharp} and B_{\sharp} are both invertible, then the scheme (4) is strongly stable if the CFL parameters λ, μ are sufficiently small.

The paper is organized as follows. In section 3, we prove Theorem 1. Strong stability is proved directly on the amplification matrix for the scheme (2) while we use the decomposition into real and imaginary parts of the amplification matrix for the scheme (3). This gives us the opportunity to give a slight refinement of the results in [11]. Section 4 is devoted to the analysis of the scheme (4) in the particular case of the matrices given in Theorem 2. We shall also address the case of invertible matrices.

3 Strong stability results

3.1 The upwind scheme

The amplification matrix of the upwind scheme (2) can be written as

$$C(\xi, \eta) = I - 2 \left(\sin^2 \frac{\xi}{2} |A| + \sin^2 \frac{\eta}{2} |B| \right) - i (\sin \xi A + \sin \eta B).$$

The sufficient condition for stability is then obtained by choosing $\xi = \eta = \pi$. We now show that this condition is also sufficient for strong stability. We decompose the amplification matrix as follows:

$$C(\xi, \eta) = I - |A| - |B| + e^{i\xi} \frac{|A| - A}{2} + e^{-i\xi} \frac{|A| + A}{2} + e^{i\eta} \frac{|B| - B}{2} + e^{-i\eta} \frac{|B| + B}{2}.$$

All five real symmetric matrices

$$I - |A| - |B|, \quad \frac{|A| - A}{2}, \quad \frac{|A| + A}{2}, \quad \frac{|B| - B}{2}, \quad \frac{|B| + B}{2},$$

are nonnegative and their sum equals the identity matrix. The conclusion follows from the following general result¹ whose proof is recalled for the sake of completeness:

Lemma 1. *Let $H_1, \dots, H_q \in \mathcal{M}_d(\mathbb{C})$ be nonnegative Hermitian matrices such that*

$$H_1 + \dots + H_q = I.$$

Then for all complex numbers z_1, \dots, z_q satisfying $|z_j| \leq 1$ for all j , there holds

$$\|z_1 H_1 + \dots + z_q H_q\| \leq 1.$$

Proof. We write

$$\begin{pmatrix} \sum z_j H_j & 0 & \dots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} = M^* \text{diag}(z_1 I, \dots, z_q I) M, \quad M := \begin{pmatrix} H_1^{1/2} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ H_q^{1/2} & 0 & \dots & 0 \end{pmatrix},$$

where $H_j^{1/2}$ denotes the unique nonnegative Hermitian square root of H_j . The assumptions on the H_j 's yield $\|M\| = \|M^*\| \leq 1$ and the result follows. \square

3.2 The spectral norm of complex symmetric matrices

The aim of this paragraph is to make a little more precise the statement of [11, Lemma 4.1] on the characterization of complex symmetric matrices with strongly stable iterates. Our result is:

Proposition 1. *Let $C \in \mathcal{M}_d(\mathbb{C})$ be a symmetric matrix that we decompose as $C = I - K + iJ$, where K and J are real symmetric matrices. Then $\|C\| \leq 1$ if and only if for all real unit vectors x, y in \mathbb{R}^d there holds*

$$\langle Kx, x \rangle \langle Ky, y \rangle + \langle Jx, y \rangle^2 \leq 2 \langle Kx, x \rangle. \quad (5)$$

Lemma 4.1 in [11] covers the "if" part of Proposition 1, and we briefly indicate why (5) gives a complete characterization of symmetric matrices in the unit ball of $\mathcal{M}_d(\mathbb{C})$.

Proof. We start with the formula (3.1) in [11]. If C is a complex symmetric matrix whose real and imaginary parts are denoted R and J (that is, $R = I - K$ with the notation of Proposition 1), there holds²

$$\|C\| = \max_{(u,v) \in \mathbb{R}^d, |u|^2 + |v|^2 = 1} \{ \langle Ru, u \rangle - \langle Rv, v \rangle + 2 \langle Ju, v \rangle \}. \quad (6)$$

¹This result seems quite classical. It appears as Exercise 301 in the additional list of exercises of [9], see <http://www.umpa.ens-lyon.fr/~serre/DPF/exobis.pdf>. It may probably be found in earlier textbooks or references that the author is not aware of.

²The equality can also be written, as in [11], with $2|\langle Ju, v \rangle|$ on the right hand side by changing u into $-u$, but this is of no consequence.

Let $x, y \in \mathbb{R}^d$ be unit vectors and let $\theta \in \mathbb{R}$. Then $u := \cos \theta x$ and $v := \sin \theta y$ satisfy $|u|^2 + |v|^2 = 1$, so (6) gives

$$\begin{aligned} 2 \|C\| &\geq 2 \cos^2 \theta \langle Rx, x \rangle - 2 \sin^2 \theta \langle Ry, y \rangle + \sin(2\theta) (2 \langle Jx, y \rangle) \\ &= \langle Rx, x \rangle - \langle Ry, y \rangle + \cos(2\theta) (\langle Rx, x \rangle + \langle Ry, y \rangle) + \sin(2\theta) (2 \langle Jx, y \rangle). \end{aligned}$$

Maximizing first over θ , and then over x, y , we get

$$2 \|C\| \geq \max_{(x,y) \in \mathbb{R}^d, |x|=|y|=1} \left\{ \langle Rx, x \rangle - \langle Ry, y \rangle + \sqrt{(\langle Rx, x \rangle + \langle Ry, y \rangle)^2 + 4 \langle Jx, y \rangle^2} \right\}.$$

The opposite inequality is shown by considering a couple $(u, v) \in \mathbb{R}^d$, $|u|^2 + |v|^2 = 1$, for which the maximum in (6) is attained. Such a couple can always be written under the form $u := \cos \theta x$, $v := \sin \theta y$, where x and y are unit vectors. Hence Corollary 3.2 in [11] can be improved as

$$2 \|C\| = \max_{(x,y) \in \mathbb{R}^d, |x|=|y|=1} \left\{ \langle Rx, x \rangle - \langle Ry, y \rangle + \sqrt{(\langle Rx, x \rangle + \langle Ry, y \rangle)^2 + 4 \langle Jx, y \rangle^2} \right\}. \quad (7)$$

Recalling [11, Lemma 4.1], it remains to prove that if C is a complex symmetric matrix whose norm does not exceed 1, then (5) holds. Let us first observe that for all real unit vector x , the complex number $\langle Cx, x \rangle$ belongs to the unit disk, so its real part $\langle Rx, x \rangle$ belongs to $[-1, 1]$. In other words, there holds $\|R\| \leq 1$. Since $\|C\| \leq 1$, we can apply (7) and obtain

$$\sqrt{(\langle Rx, x \rangle + \langle Ry, y \rangle)^2 + 4 \langle Jx, y \rangle^2} \leq 2 - \langle Rx, x \rangle + \langle Ry, y \rangle,$$

for all unit vectors x, y (the right hand side is nonnegative since $\|R\| \leq 1$). Squaring the inequality and simplifying some terms, we obtain

$$\langle Rx, x \rangle^2 + \langle Jx, y \rangle^2 \leq 1 - (\langle Rx, x \rangle - \langle Ry, y \rangle)(1 - \langle Rx, x \rangle),$$

which is nothing but (5) by recalling $R = I - K$. □

3.3 The Lax-Wendroff scheme with stabilizer

Our goal is now to prove that the Lax-Wendroff scheme with stabilizer (3) is stable, and even strongly stable, if and only if $2(A^2 + B^2) \leq I$. The amplification matrix of (3) reads $C(\xi, \eta) = I - K + iJ$ with

$$\begin{aligned} J &:= \sin \xi A + \sin \eta B, \\ K &:= \frac{1}{2} J^2 + 2 \left(\sin^2 \frac{\xi}{2} + \sin^2 \frac{\eta}{2} \right) \left(\sin^2 \frac{\xi}{2} A^2 + \sin^2 \frac{\eta}{2} B^2 \right). \end{aligned} \quad (8)$$

Following [8], we choose $\xi = \eta = \pi$ and find that the condition $2(A^2 + B^2) \leq I$ is necessary for stability. Let us now show that it is also sufficient for proving *strong stability* of (3). In view of Proposition 1, see also [11, Corollary 4.3], it is sufficient to show that for all real unit vectors x, y in \mathbb{R}^d there holds³

$$\langle Kx, x \rangle \langle Ky, y \rangle \leq \langle (2K - J^2)x, x \rangle. \quad (9)$$

We use the definition (8) and thus wish to show the inequality

$$\langle Kx, x \rangle \langle Ky, y \rangle \leq 4 \left(\sin^2 \frac{\xi}{2} + \sin^2 \frac{\eta}{2} \right) \left(\sin^2 \frac{\xi}{2} |Ax|^2 + \sin^2 \frac{\eta}{2} |By|^2 \right). \quad (10)$$

³This inequality implies (5) thanks to Cauchy-Schwarz.

For simplicity, we use the notation

$$\alpha := \sin^2 \frac{\xi}{2}, \quad \beta := \sin^2 \frac{\eta}{2}.$$

We estimate $\langle K y, y \rangle$ for all real unit vector y . The matrix K reads

$$K = 2\alpha A^2 + 2\beta B^2 + 2\alpha\beta(A^2 + B^2) + \frac{1}{2} \sin \xi \sin \eta (AB + BA).$$

Since $|Ay|^2 + |By|^2 \leq 1/2$, we can write

$$|Ay|^2 = \frac{r}{2}(1 + \sin \theta), \quad |By|^2 = \frac{r}{2}(1 - \sin \theta), \quad 0 \leq r \leq \frac{1}{2}, \quad |\theta| \leq \frac{\pi}{2}.$$

We compute

$$\begin{aligned} \langle K y, y \rangle &= r(\alpha + \beta) + 2r\alpha\beta + r \sin \theta (\alpha - \beta) + \sin \xi \sin \eta \langle Ay, By \rangle \\ &\leq r(\alpha + \beta + 2\alpha\beta) + r \sin \theta (\alpha - \beta) + |\sin \xi| |\sin \eta| |Ay| |By|, \end{aligned}$$

and we thus derive the estimate

$$\begin{aligned} \langle K y, y \rangle &\leq r(\alpha + \beta + 2\alpha\beta) + r \left\{ \sin \theta (\alpha - \beta) + \cos \theta \frac{|\sin \xi| |\sin \eta|}{2} \right\} \\ &\leq r(\alpha + \beta + 2\alpha\beta) + r \left\{ (\alpha - \beta)^2 + \frac{\sin^2 \xi \sin^2 \eta}{4} \right\}^{1/2}. \end{aligned}$$

It remains to compute

$$\left\{ (\alpha - \beta)^2 + \frac{\sin^2 \xi \sin^2 \eta}{4} \right\}^{1/2} = \{(\alpha + \beta - 2\alpha\beta)^2\}^{1/2} = \alpha + \beta - 2\alpha\beta,$$

where the last equality comes from the fact that $\alpha, \beta \in [0, 1]$. Eventually we have derived the estimate

$$\langle K y, y \rangle \leq 2r(\alpha + \beta) \leq \alpha + \beta.$$

Coming back to (10), we see that it only remains to show

$$\langle K x, x \rangle \leq 4(\alpha |Ax|^2 + \beta |Bx|^2),$$

which is precisely what is obtained in [11, page 75]. Let us recall the derivation of this final estimate for the sake of clarity. The definition (8) gives

$$\langle K x, x \rangle = \frac{1}{2} |Jx|^2 + 2(\alpha + \beta)(\alpha |Ax|^2 + \beta |Bx|^2).$$

The norm of the vector Jx is estimated as follows:

$$\begin{aligned} |Jx|^2 &\leq \sin^2 \xi |Ax|^2 + \sin^2 \eta |Bx|^2 + 2 |\sin \xi| |\sin \eta| |Ax| |Bx| \\ &= 4\alpha(1 - \alpha) |Ax|^2 + 4\beta(1 - \beta) |Bx|^2 + 8\sqrt{\alpha(1 - \beta)\beta(1 - \alpha)} |Ax| |Bx| \\ &\leq 4\alpha(2 - \alpha - \beta) |Ax|^2 + 4\beta(2 - \alpha - \beta) |Bx|^2, \end{aligned}$$

which eventually gives the expected estimate:

$$\langle K x, x \rangle \leq 4(\alpha |Ax|^2 + \beta |Bx|^2).$$

We have thus shown that (10) holds, and Proposition 1 shows that the scheme (3) is strongly stable.

4 The Lax-Wendroff scheme without stabilizer

Let us first recall that the scheme (4) is known to be stable under the condition that for all real unit vector u , there holds $|Au|^{2/3} + |Bu|^{2/3} \leq 1$, see [12]. (The reader can also consult [5] for the more restrictive - though easier - criterion $|Au|^2 + |Bu|^2 \leq 1/4$.)

We first study the finite difference scheme (4) when the matrices A_\sharp and B_\sharp are given by:

$$A_\sharp := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_\sharp := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The amplification matrix of the Lax-Wendroff scheme (4) reads $C(\xi, \eta) = I - K + iJ$ with

$$K := 2 \sin^2 \frac{\xi}{2} A^2 + 2 \sin^2 \frac{\eta}{2} B^2 + \frac{1}{2} \sin \xi \sin \eta (AB + BA), \quad J := \sin \xi A + \sin \eta B. \quad (11)$$

Let us assume that the scheme is strongly stable. In particular, for all $\eta \in \mathbb{R}$, there holds $\|C(\pi, \eta)\| \leq 1$. Choosing $x := e_2$ and $y := e_1$ the vectors that span the canonical basis of \mathbb{R}^2 , Proposition 1 gives

$$\langle e_2, K e_2 \rangle \langle e_1, K e_1 \rangle + \langle J e_2, e_1 \rangle^2 \leq 2 \langle e_2, K e_2 \rangle,$$

that is,

$$\left(2\mu^2 \sin^2 \frac{\eta}{2}\right) \left(2\lambda^2 + 2\mu^2 \sin^2 \frac{\eta}{2}\right) + \mu^2 \sin^2 \eta \leq 4\mu^2 \sin^2 \frac{\eta}{2}.$$

The latter condition reads

$$\mu^2 \sin^2 \frac{\eta}{2} \left(\lambda^2 + \mu^2 \sin^2 \frac{\eta}{2}\right) \leq \mu^2 \sin^4 \frac{\eta}{2},$$

which gives obviously a contradiction by choosing $\eta > 0$ small enough.

Let us now assume that the matrices A_\sharp and B_\sharp are invertible, and let us prove that for sufficiently small CFL parameters λ, μ the amplification matrix defined by (11) has strongly stable iterates. Our goal is to show that the criterion (9), which is sufficient for strong stability, is satisfied for λ, μ small enough. We still use the notation

$$\alpha := \sin^2 \frac{\xi}{2}, \quad \beta := \sin^2 \frac{\eta}{2}.$$

From the definitions (11), we compute

$$\langle (2K - J^2)x, x \rangle = 4\alpha^2 |Ax|^2 + 4\beta^2 |Bx|^2 \geq c(\alpha^2 \lambda^2 + \beta^2 \mu^2),$$

where we have used the invertibility of A_\sharp and B_\sharp , and the constant $c > 0$ is independent of $\alpha, \beta, \lambda, \mu$ and x . Let us now give a bound for $\|K\|$:

$$\begin{aligned} \|K\| &\leq 2\alpha\lambda^2 \|A_\sharp\|^2 + 2\beta\mu^2 \|B_\sharp\|^2 + |\sin \xi| |\sin \eta| \lambda\mu \|A_\sharp\| \|B_\sharp\| \\ &\leq C \left(\alpha\lambda^2 + \beta\mu^2 + 2\sqrt{\alpha\beta}\lambda\mu \right) \\ &\leq C \max(\lambda, \mu) (\sqrt{\alpha\lambda} + \sqrt{\beta\mu})^2, \end{aligned}$$

where C denotes a positive constant, which may vary from one line to the next, that does not depend on $\alpha, \beta, \lambda, \mu$. Hölder's inequality yields

$$\|K\|^2 \leq C \max(\lambda^2, \mu^2) (\alpha^2 \lambda^2 + \beta^2 \mu^2),$$

so choosing λ, μ small enough, the inequality (9) is satisfied. This tends to indicate that if one restricts to numerical schemes that are dissipative in Kreiss's sense, then strong stability

might be recovered for sufficiently small CFL parameters. This assertion should be taken very cautiously though, and we have no other example of this fact except (4).

Eventually, let us observe that strong stability can hold for (4) with noncommuting matrices, even when the CFL parameters are not small. Let us consider for instance the case

$$A_{\sharp} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_{\sharp} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which corresponds to the two-dimensional wave equation. Then the amplification matrix of (4) reads

$$\left(1 - 2\lambda^2 \sin^2 \frac{\xi}{2} - 2\mu^2 \sin^2 \frac{\eta}{2}\right) I - i(\sin \xi \lambda A_{\sharp} + \sin \eta \mu B_{\sharp}).$$

This is a normal matrix whose spectral radius is not larger than 1 for all (ξ, η) if and only if $\lambda^2 + \mu^2 \leq 1$. In that case, stability is equivalent to strong stability. As expected from the general theory in [12], the stability domain encompasses the "scalar" one ($\lambda^{2/3} + \mu^{2/3} \leq 1$).

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